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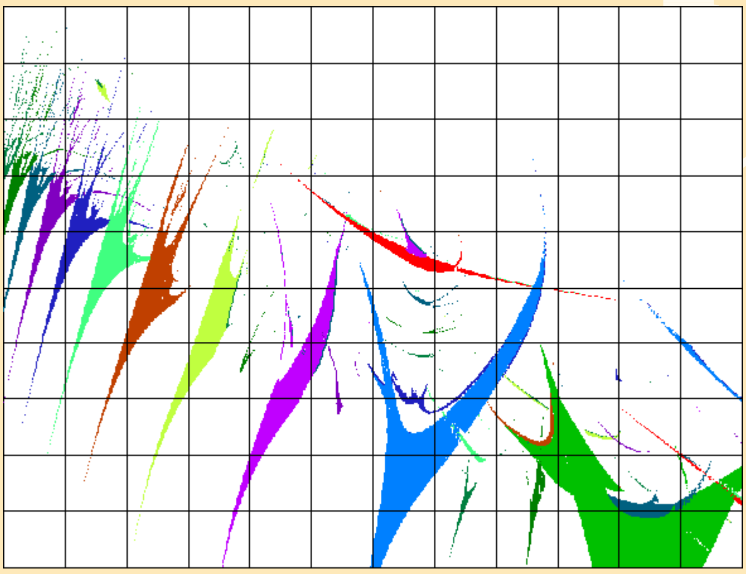

CERUM Working Paper 47:2002

A Business Cycle Model with Cubic Nonlinearity

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Abstract: This paper deals with a simple multiplier-accelerator model of the business cycle, including a cubic nonlinearity. The corresponding two dimensional iterative map is represented in terms of its bifurcation diagram in parameter space. A number of bifurcation sequences for attractors and their basins are studied.

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CERUM Working Paper 47:2002
ISBN 91-7305-289-2
ISSN 1404–5362

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Introduction

The focus of the present exposition is visualization techniques. For this a simple nonlinear business cycle model of multiplier-accelerator type is employed. There have been presented numerous most interesting contributions to business cycle modelling during the last decades, using more complicated relations, and involving for instance the monetary sector of the economy in a more general Keynesian setting.

It is, however, interesting to see how complex the scenarios can become with even the most simplified assumptions, to which complications of the model can only add even more complexity.

The procedure is as follows. We first outline the model studied, and relate it to the historical roots, including an outline of some early results for particular cases.

Then follows a study of the fixed, or equilibrium point and its stability. The Neimark-Sacker bifurcation of the fixed point is described in detail. Next, a reference diagram in the parameter plane, Figure 3, presents information on the stability region for the fixed point, the region of divergence to infinity, and the existence regions for attracting cycles of different periods. Then two particular routes of successive bifurcations due to parameter changes are studied in some detail, and the resulting attractors and basins (in cases of coexistence) are shown.

The Linear Multiplier-Accelerator Model

One of the first formal mathematical models for business cycles was due to Paul Samuelson (1939). He combined the multiplier, recently emergent with Keynesian macroeconomics, with the older "principle of acceleration", to formulate something which essentially worked as a simple harmonic oscillator. The main shortcoming, provided one wanted to explain sustained cyclic change in the economy, was that, except for one nongeneric boundary case, either movement would go to extinction, or the system would explode. This always is the case with linear oscillators. Ragnar Frisch (1933) argued that all systems be damped, but be kept going through exogenous shocks. Sir John Hicks (1950) preferred the explosive case, but introduced bounds ("floor" and "roof"), which not only removed the absurdities of explosion, but actually made the system piecewise linear, i. e. nonlinear.

Let us so pin down the basics of the Samuelson-Hicks model. Productive capital was assumed to be held in proportion to output (i. e. income in real terms), so investment, being the change in capital stock, was in proportion to income change:

$$I_t = v\,(Y_{t-1} - Y_{t-2}) \tag{1}$$

with I_t denoting investment and $(Y_{t-1} - Y_{t-2})$ denoting the change of income between the two previous periods. The constant capital to output ratio v was the "accelerator". Likewise, consumption was a given proportion of income in the previous period:

$$C_t = (1 - s)\,Y_{t-1} \tag{2}$$

where $0 \leq s \leq 1$ was the complementary proportion saved, and $1/s$ was the Keynesian multiplier. Introducing the accounting identity for a closed economy:

$$Y_t = C_t + I_t \tag{3}$$

we can derive a simple recurrence equation in the income variable only, namely:

$$Y_t = (1 + v - s)\,Y_{t-1} - vY_{t-2} \tag{4}$$

For any nonzero initial condition, equation (4) will generate persistent oscillations only for very special parameter values, i. e., $v = 1$ and $0 \leq s \leq 4$, which is implied by $0 \leq s \leq 1$. In this case, the amplitude of oscillations depends entirely on initial conditions. Notice that (4) is somewhat different from Samuelson's own model in which the accelerator was applied only to consumption, but, details apart, the formulations are equivalent.

The Nonlinear Accelerator

As mentioned, Hicks (1950) assumed there were upper and lower bounds for investment $I_t < I_{\max}$, and $I_t > I_{\min}$. The latter bound was negative and represented the disinvestment when no capital at all was replaced but depreciated at its natural rate. Hicks's objection to the linear investment function was mainly due to the fact that an unbounded linear investment function would imply active destruction of capital in depression phases of the business cycle. Goodwin (1951) proposed that the limits be approached asymptotically, and the complete investment function take the form of a hyperbolic or an arc tangent type. He was able to show that the system then went to a limit cycle attractor.

In Puu (1989) a linear-cubic shape for the investment function was proposed as an alternative. This had the additional feature of being back-bending. As a representation of the total investment, including the public sector, this can include the realistic fact that governments tend to distribute infrastructure investment contracyclically, partly as a means to fight depressions, partly as a means to profit from lower input prices during slumps.

The investment function hence became:

$$I_t = v\left(Y_{t-1} - Y_{t-2}\right) - v\left(Y_{t-1} - Y_{t-2}\right)^3 \tag{5}$$

Hicks's original model also included autonomous expenditures, by the public sector and such by the private sector which were not cycle dependent. As is always the case with linear systems, the autonomous expenditures just result in a positive stationary value (these expenditures scaled up by the multiplier $1/s$), so that Y_t has the meaning of a deviation from this stationary income, taking both positive and negative values. We note that the inclusion of a cubic term does not make any difference at all in this respect.

Further in Puu (1989) the consumption function was a bit different from what Hicks assumed:

$$C_t = (1 - s)\, Y_{t-1} + sY_{t-2} \tag{6}$$

In the original setup incomes saved were assumed to be withdrawn from consumption for eternity, whereas in the above function they were assumed to be kept for just one period, and totally consumed one period later, hence the two lagged contributions to consumption.

Again, substituting in the accounting identity $Y_t = C_t + I_t$ we now, using a slight rearrangement of terms, have:

$$Y_t - Y_{t-1} = (v - s)\left(Y_{t-1} - Y_{t-2}\right) - v\left(Y_{t-1} - Y_{t-2}\right)^3 \tag{7}$$

so introducing the new variable $Y_t - Y_{t-1} = Z_{t-1}$, we can restate the system as:

$$Y_t = Y_{t-1} + Z_{t-1} \tag{8}$$

$$Z_t = (v - s)\, Z_{t-1} - vZ_{t-1}^3 \tag{9}$$

As we see the last equation is autonomous in Z_t, income difference, whereas the first just tells us how income is obtained as a successive sum of these income differences. This was the great advantage of assuming saving to be for just one

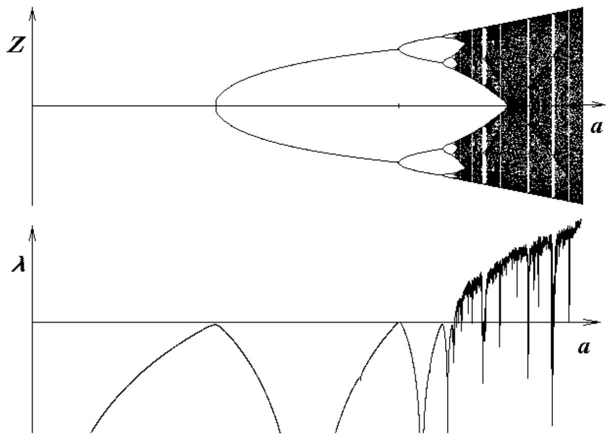

Figure 1: Bifurcation diagram and Lyapunov exponent for the autonomous cubic iteration.

period. The assumption, admittedly, was special, but no more then the original one. Later we will consider a more general case where a fraction of the savings are spent and the rest is retained for ever. Rescaling both variables by the factor $\sqrt{(1+v-s)/v}$ transforms equation (9) into:

$$Z_t = aZ_{t-1} - (a+1)\, Z_{t-1}^3 \tag{10}$$

where $a = (v - s)$. The propensity to save is less than unity, $s < 1$, whereas all past empirical measurements of the accelerator resulted in $v \gg 1$, so we can take $a > 0$.

The results of this iteration, obtained in Puu (1989), are summarized in Figure 1, in terms of a bifurcation diagram on top of the picture and a curve for the Lyapunov exponent below. We see a scenario going from a zero equilibrium to two coexistent nonzero equilibria $Z = \pm\sqrt{(a-1)/(a+1)}$, which split in coexistent two-cycles at $a = 2$, after which there follow period doubling cascades to chaos. At a value of $a = \frac{3}{2}\sqrt{3}$ the chaotic attractors are no longer separate, but merge in one single attractor.

A More General Consumption Function

A more general case was studied in Puu (1991), where the consumption function took the form:

$$C_t = (1-s)\, Y_{t-1} + \varepsilon s Y_{t-2} \tag{11}$$

A fraction $0 \le \varepsilon \le 1$ of savings was assumed to be spent after being saved for one period. So, for $\varepsilon = 1$ the previous case is recovered, whereas $\varepsilon = 0$ corresponds to the original Hicks case.

Substituting from equations (5) and (11) into (3), again using the rescaling $\sqrt{(1+v-s)/v}$ of variables, we can put the system in the form:

$$\begin{aligned} Y_t &= Y_{t-1} + Z_{t-1} \\ Z_t &= aZ_{t-1} - (a+1)\, Z_{t-1}^3 - bY_{t-1} \end{aligned} \tag{12}$$

where $b = (1-\varepsilon)s$ represents a sort of eternal rate of saving. The equations are now no longer uncoupled, and hence have a geometry in 2-dimensional phase space. Figure 2 displays a series of pictures where we keep $a = 2$, but let b decrease from 0.125, through 0.05 and 0.01, to a vanishing value. We see how the normal shape of chaotic attractor from upper left to lower right transforms into a relaxation cycle in the terminology of the perturbation literature, i. e. a combination of smooth

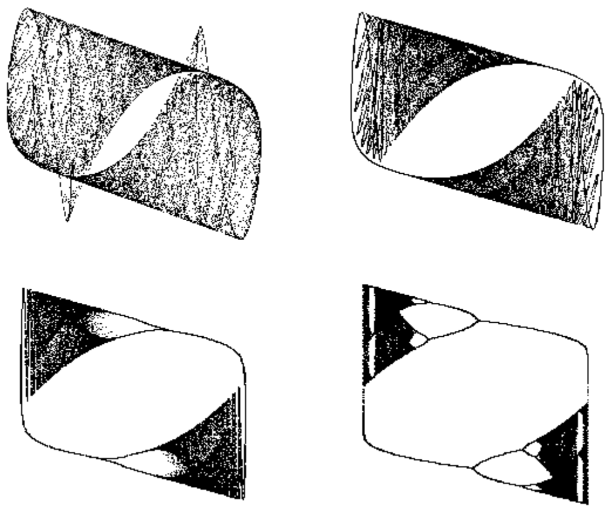

Figure 2: Emergence of the relaxation cycle for the coupled system.

movement and sudden jumps, further with two inserted copies of the bifurcation diagram from Figure 1.

In Puu (1991) mainly this scenario $b \to 0$ was studied. Define a new variable $X_t = bY_t$, form the quotient $(Z_t - Z_{t-1}) / (X_t - X_{t-1})$ from (12), and divide by b. Thus:

$$\frac{(Z_t - Z_{t-1})}{(X_t - X_{t-1})} = \frac{1}{b} \frac{(a-1) Z_{t-1} + (a+1) Z_{t-1}^3 - X_{t-1}}{Z_{t-1}} \tag{13}$$

As we see, the right hand side goes to infinity when $b \to 0$, i. e. trajectories become vertical in the X, Z phase space, except when the cubic in the numerator is zero too. Deleting indices, this means that:

$$(a-1) Z - (a+1) Z^3 = X \tag{14}$$

This so called characteristic, in the terminology of perturbation studies, is a curve in X, Z-phase space along which the system can move in directions other than the vertical. Whenever this lying cubic turns around, the process must drop or jump to a relevant unique portion of the curve. Such a cycle was called a relaxation cycle in perturbation studies. We see it in the last picture of Figure 2, along with two copies of the bifurcation diagram from Figure 1 inserted. The resulting process is a cycle where the relaxation jump enters a chaotic zone from which the cycle again simplifies through a *period halving route to order*. In terms of very rough realism this could seem reasonable, as the disordered phases of the cycle would occur in the transitions between phases of prosperity and depression.

This is but one scenario for the model. In the following we will study it in more general cases, considering its bifurcations and various periodic, chaotic, and explosive regimes.

Dynamic Behavior of the Model

The question we discuss in this section concerns the different kinds of attracting sets and their parameter dependent bifurcations, which show up when we study the model (12) in its dynamic context, and even lead to chaos

Note that the system (12) is noninvertible, so global analyses which use the theory of critical lines (Mira, 1987, Mira, *et al.* 1996) apply. Noninvetibility means that there exists a set in phase space where the Jacobian determinant of the map vanishes. The forward image of this set is called critical line LC. The existence of such a set brings a specific character into bifurcation scenarios, shapes of attracting sets and their basins of attraction, etc., different from those known for invertible maps.

Fixed point bifurcation

Let us now change notation in the model (12): $x := Y$, $y := Z$. Thus, we consider a dynamical system generated by a family of two-dimensional continuous noninvertible maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ ay-(a+1)y^3-bx \end{pmatrix}, \tag{15}$$

where a, b are real parameters such that $a > 0$, $0 < b < 1$.

The map F, obviously, has the single fixed point $(x_0, y_0) = (0, 0)$. Let us derive the stability conditions for this (x_0, y_0). The Jacobian matrix of F depends only on y and has the form

$$DF = \begin{pmatrix} 1 & 1 \\ -b & a-3(a+1)y^2 \end{pmatrix}. \tag{16}$$

Corresponding eigenvalues at the fixed point $(0, 0)$ are equal to

$$\mu_{1,2} = (a+1 \pm \sqrt{(a-1)^2-4b})/2.$$

One can easily get a triangle S of stability of the fixed point $(0, 0)$ in the (b, a)-parameter plane for which $|\mu_{1,2}| < 1$. Necessary and sufficient conditions for stability are the following inequalities:

$$\begin{cases} 1+|DF|+trDF > 0, \\ 1+|DF|-trDF > 0, \\ 1-|DF| > 0, \end{cases}$$

where $|DF|$ and $trDF$ denote the determinant and the trace of DF, respectively. The above inequalities for the Jacobian matrix (16) at the fixed point $(0, 0)$ become

$$\begin{cases} 2+2a+b > 0, \\ b > 0, \\ 1-a-b > 0. \end{cases}$$

Taking into account that the map F is defined only for $a > 0$ and $0 < b < 1$, the triangle S is given by

$$S = \{(b, a) : 0 < b < 1,\ 0 < a < 1-b\}.$$

Let $(b, a) \in S$. Then the fixed point $(0, 0)$ is stable. It is a stable focus because $\mu_{1,2}$ are complex conjugate for $1-2\sqrt{b} < a < 1+2\sqrt{b}$. One can easily see that this inequality is fulfilled for the triangle S. We can write $\mu_{1,2}$ as $\mu_{1,2} = \mathrm{Re}\,\mu_{1,2} \pm i\,\mathrm{Im}\,\mu_{1,2}$, where

$$\mathrm{Re}\,\mu_{1,2} = \frac{a+1}{2};\ \mathrm{Im}\,\mu_{1,2} = \frac{\sqrt{(a-1)^2-4b}}{2}.$$

Let us now study the bifurcation of stability loss of the fixed point, which occurs at

$$a = 1 - b \tag{17}$$

when the eigenvalues cross the unit circle, i. e. $|\mu_{1,2}| = 1$. It is known (see, e. g., Sacker, 1964, Ioss, 1979, Mira, 1987) that if there is no so-called strong resonance 1:k, $k \leq 4$, that is

$$\mathrm{Re}\,\mu_{1,2} \neq \cos\frac{2\pi l}{k},\ k \leq 4,$$

then a Neimark-Sacker bifurcation occurs and results in the appearance of an attracting invariant closed curve (homeomorphic to a circle, in other words, a two-dimensional torus) in the neighborhood of the fixed point. The map is reduced to a rotation map on this curve. It can be shown that for the parameter range considered strong resonance cannot occur, given the condition $a > 0$. Indeed, the resonance 1:2 occurs at $a = -3$, when $\mu_1 = \mu_2 = -1$, the resonance 1:3 at $a = -2$, and 1:4 at $a = -1$. The case $k = 1$ occurs at $a = 1$, $b = 0$, when $\mu_1 = \mu_2 = 1$.

The invariant circle can have a rational or an irrational rotation number, depending on the parameters. In the case of a *rational* rotation number l/k, where l/k is an irreducible fraction, this invariant circle consists of the unstable manifold of a saddle cycle of period k approaching points of an attracting cycle of the same period. The value k depends on a resonance region in which the eigenvalues have particular values after crossing the unit circle (Arnold *et al.*, 1986). Such regions reach the unit circle at points $e^{2\pi il/k}$ in the of form of densely packed narrow tongues. Thus, in the general case the eigenvalues cross a countable number of the tongues near the unit circle. In the parameter space it corresponds to the case when the parameter point crosses, near the bifurcation curve, the so-called "Arnold tongues" associated with the attracting cycles of different periods. Note that the map F is symmetric with respect to 0. Thus a cycle of any odd period, by necessity asymmetric, must then appear together with one more cycle of the same period, such that they together provide for symmetry.

In the case of an *irrational* rotation number, typical trajectories of the map are everywhere dense on the invariant circle. The probability for the eigenvalues to cross the unit circle at an irrational point is obviously larger then at a rational point (because the measure of the set of rational numbers is equal to 0) but soon after the bifurcation it is more probable to get an invariant circle with a rational rotation number, i. e., an attracting cycle of rather high period. On a generic one-dimensional curve crossing the parameter plane close to the bifurcation curve, the sequence of the regions, corresponding to the invariant circles with rational/irrational rotation numbers, is described by the behavior of the Devil's staircase (reported in many textbooks), where "intervals" are associated with rational rotations.

We can easily obtain the values a_k and b_k of the parameters a and b, such that an invariant circle with rotation number l/k appears after stability loss of the fixed point. In other words, we get the parameter values when attracting and saddle cycles of period k are born (two pairs in the case of an odd period). This can be obtained from the following equality:

$$\mathrm{Re}\,\mu_{1,2} = \cos\frac{2\pi l}{k},\ k > 4,$$

from which it follows that corresponding value of the parameter a is

$$a_k = 2\cos\frac{2\pi l}{k} - 1,$$

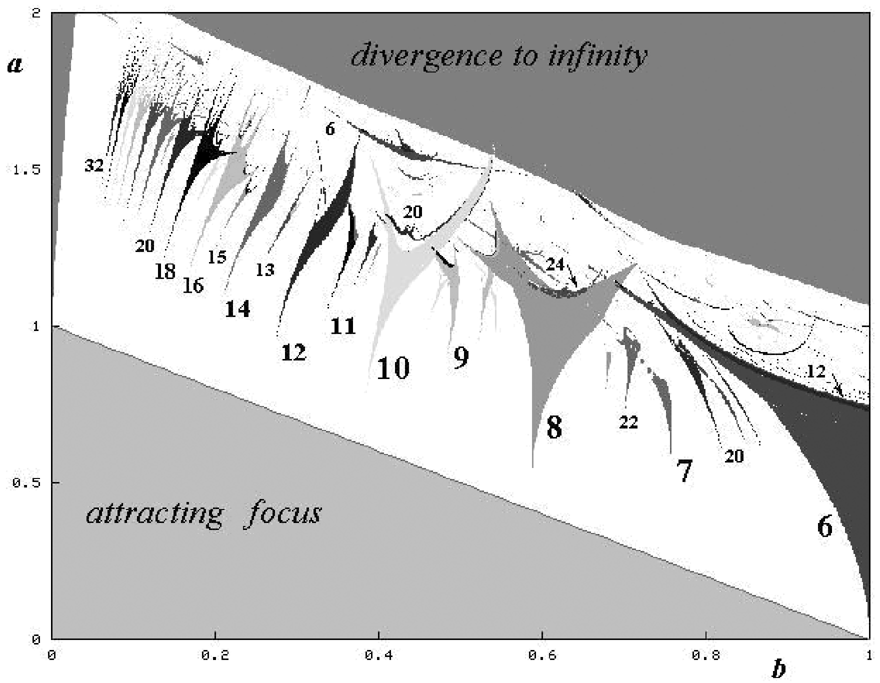

Figure 3: The two-dimensional bifurcation diagram of the map F in the (b, a)-parameter plane where the existence regions of attracting cycles of the period k, $k \leq 32$, are shown. The periods are indicated by corresponding numbers.

and from (17) it follows that corresponding value of the parameter b is

$$b_k = 1 - a_k.$$

Thus, for instance, an attracting cycle of period $k = 6$ appears together with a saddle cycle of period 6 at $a = 0$, $b = 1$. Two attracting and two saddle period-7 cycles appear at $a \approx 0.24698$, $b \approx 0.75302$. Attracting and saddle period-8 cycles appear at $a = \sqrt{2} - 1$, $b = 2 - \sqrt{2}$, and so on.

Figure 3 presents the two-dimensional bifurcation diagram in the (b, a)-parameter plane where the regions of existence of attracting cycles of periods k, $k \leq 32$, and the region of divergence to infinity are shown. The above mentioned "Arnold tongues" are clearly seen. An enlarged window of this diagram is shown in Figure 4 where the regions of coexistence of several attracting cycles can be observed. These occur where there are overlapping tongues associated with cycles of different periods. The white region corresponds to the parameter values such that an attracting set of the map F is either some cycle of period larger than 32, or a chaotic attractor, or an invariant circle with irrational rotation number. The attracting sets of all types can be coexisting.

Critical lines and absorbing area

Before the description of transitions to chaos we define the critical lines and the absorbing area (see Mira, *et al.*, 1996) for the map F. A train of critical lines, tangent to each other, bounds the absorbing area for the attractor. The initial line, said to have rank –1, is obtained by equating the Jacobian determinant to zero. It is iterated a sufficient number of times to completely bound the absorbing area, each forward iterate taking a rank number higher by one. One starts numbering by zero (or dropping the index) from the first iterate because its preimage does not yet bound the absorbing area, but intersects it.

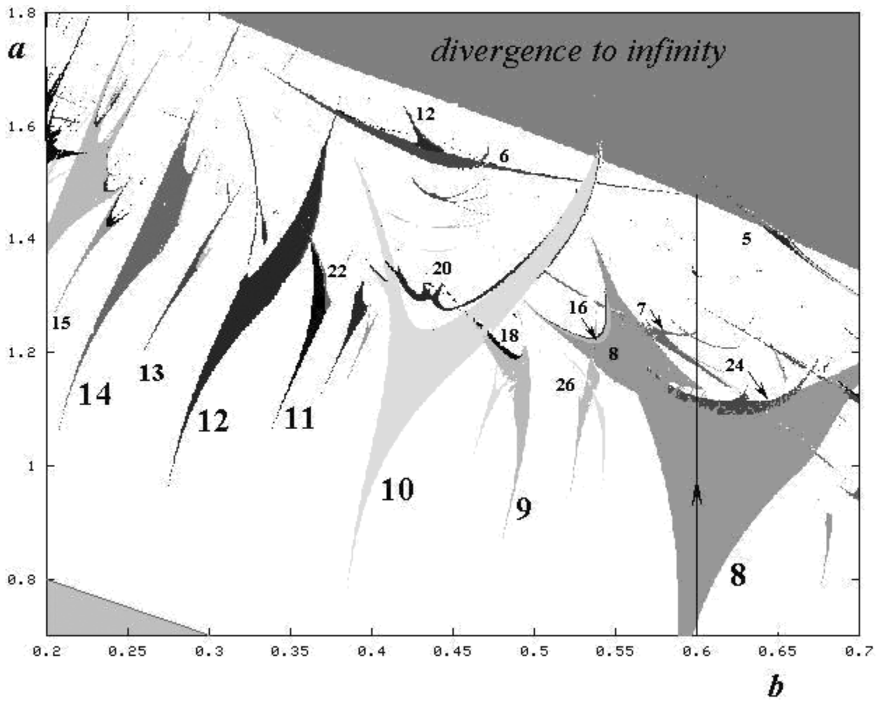

Figure 4: An enlarged window of Figure 3.

The critical line of rank –1 of the map F is obtained as the locus of points such that $|DF| = 0$ which yields two horizontal straight lines:

$$\{LC_{-1}, LC'_{-1}\} = \left\{(x, y) \in \mathbb{R}^2 : y = \pm\sqrt{\frac{a+b}{3(a+1)}}\right\}.$$

The first iteration of LC_{-1} and LC'_{-1} gives the critical lines LC and LC':

$$\{LC, LC'\} = \left\{(x, y) \in \mathbb{R}^2 : y = -bx \pm \frac{2(a+b)}{3}\sqrt{\frac{a+b}{3(a+1)}}\right\}.$$

The critical lines of rank k are defined as $LC_k = F^k(LC)$ and $LC'_k = F^k(LC')$. Note that, due to the symmetry of the map F, the set of lines LC_k and LC'_k are mutually symmetric with respect to 0.

The successive images of the critical lines define an absorbing area $\mathcal{A}$ such that if some trajectory enters $\mathcal{A}$ it can never leave this area, and such that there exists a neighborhood $U(\mathcal{A})$ whose points will be mapped into $\mathcal{A}$ in a finite number of iterations. The boundary $\partial\mathcal{A}$ is made up by portions of the images of LC and LC'. If we denote $\mathcal{A} \cap LC_{-1}$ by $[b_0, a_0]$ and $\mathcal{A} \cap LC'_{-1}$ by $[b'_0, a'_0]$, then for a suitable integer m,

$$\partial\mathcal{A} = \bigcup_{k=1}^{m} F^k([b_0, a_0]) \bigcup_{k=1}^{m} F^k([b'_0, a'_0]),$$

which obviously is symmetric as well with respect to 0 set.

Figure 5 shows an example of the absorbing area together with an attractor of the map F at $a = 0.8$, $b = 1$, where the absorbing area is made up by 6 pieces of critical lines. Some other examples of the absorbing area with attractors and their basins of attractions for the map F are illustrated in (Puu, 2000).

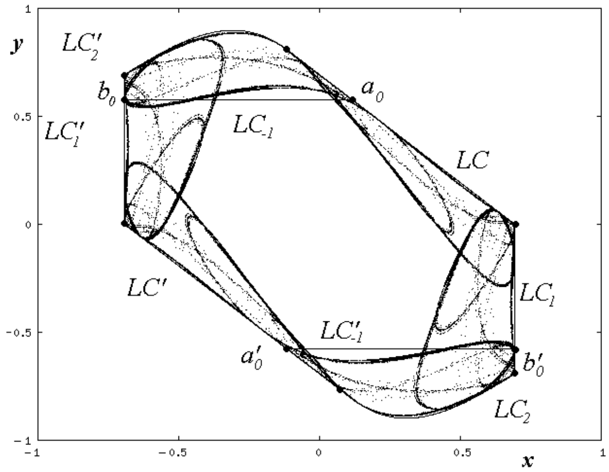

Figure 5: An example of the chaotic attractor and the absorbing area of the map F at $a = 0.8$, $b = 1$.

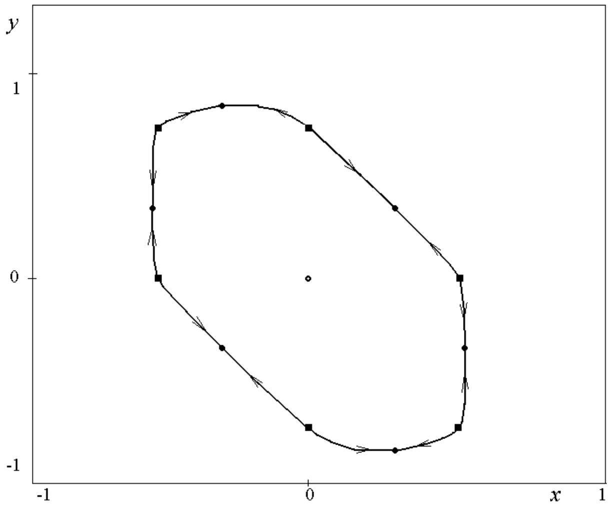

Figure 6: The invariant circle with rotation number 1/6 of the map F at $a = 0.3$, $b = 1$. The points of the saddle period-6 cycle are shown by squares and of the attracting period-6 cycle by circles.

Several scenarios of transition to chaos

What will happen with the phase portrait of the system if we fix the parameter values inside some tongue of periodicity (see Figure 3) and increase a? This question is closely related to the problem of the breakdown of a two-dimensional torus, which is well studied for diffeomorphisms (see, e. g., Afraimovich & Shil'nikov, 1983), but is a more complicated task for noninvertible maps (see, e. g., Mira, 1987, Gardini, *et al.*, 1994).

In this subsection we consider two different routes to chaos. The first is connected to the period doubling cascade of the attracting period-6 cycle, which appears after stability loss of the fixed point at $a = 0$, $b = 1$. The second is related to a more complicated sequence of bifurcations of the attracting period-8 cycle, which bifurcates from the fixed point at $a = \sqrt{2} - 1$, $b = 2 - \sqrt{2}$.

Let us fix $b = 1$, $a = 0.3$ (this parameter point is inside the 6-tongue of Figure 3), and increase the value of a. The only attractor of the map F is the attracting period-

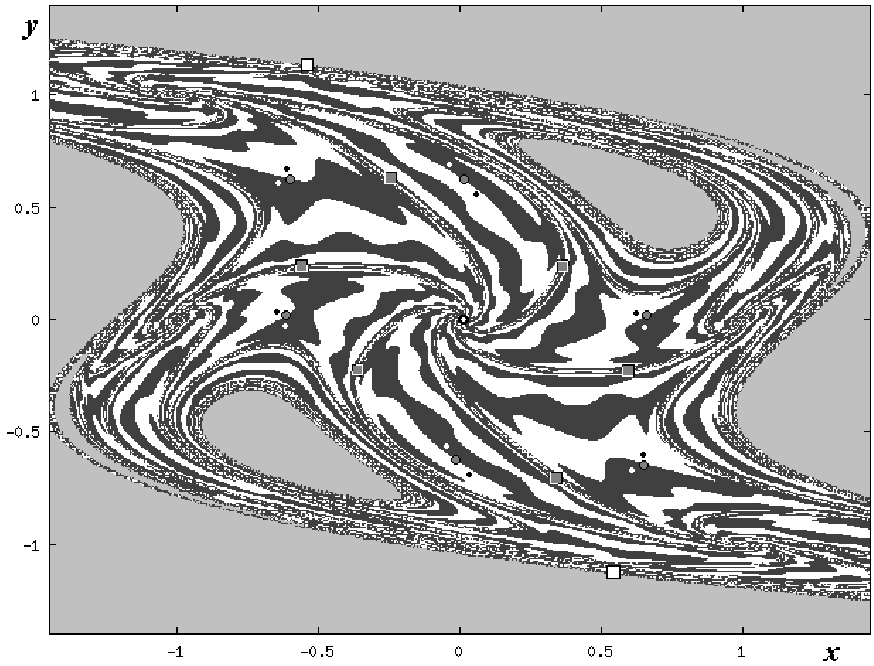

Figure 7: Two attracting period-6 cycles (the black and white circles) together with their basins of attraction. The 1^{st} 6-saddle is indicated by the grey circles, the 2^{nd} 6-saddle by the grey squares and the 2-saddle by the white squares. The grey area is the basin of infinity. Here $a = 0.68$, $b = 1$.

6 cycle, shown in Figure 6 together with the saddle period-6 cycle (we call it the 1^{st} 6-saddle), and its unstable manifold, which forms an invariant circle.

At $a \approx 0.667$ the attracting period-6 cycle undergoes a pitchfork bifurcation (its eigenvalue passes through 1): this period-6 cycle becomes a period-6 saddle (called the 2^{nd} 6-saddle) surrounded by two new attracting period-6 cycles. The immediate basin boundary of the attracting period-6 cycles is formed by the stable manifolds of the 2^{nd} 6-saddle and its preimage. This boundary approaches in the limit the stable manifold of the 1^{st} 6-saddle. The boundary of infinity (i. e., the region of points which diverge to infinity under iteration by the map F) is formed by the stable manifold of a saddle period-2 cycle (see Figure 7) which always exists.

Next bifurcations under increasing a are cascades of period doubling bifurcations of the two attracting period-6 cycles: the first period doubling occurs at $a \approx 0.725$ (one of the eigenvalues of each period-6 cycle passes through –1), the second at $a \approx 0.742$, the third at $a \approx 0.746$, and so on. Note that the period doubling cascade is realized on some one-dimensional manifold that is impossible for invertible maps: on the way an eigenvalue of some cycle, from 1 (when the cycle is born) to –1 (the period-doubling bifurcation), this eigenvalue cannot pass through 0. Thus, to avoid 0 the eigenvalues move into complex plane, so that some rotation in the phase space arises and the dynamics becomes two-dimensional.

After the cascade of the period doubling bifurcations, the resulting attracting set of the map F consists of two period-6 chaotic attractors (see Figure 8), which merge into a one-piece chaotic attractor at $a \approx 0.7925$ (see Figure 5), due to the contact with the immediate basin boundary. This merging is accompanied by the so-called "rare points phenomenon" (Maistrenko, *et al.*, 1998), due to the fractal structure of the basin boundary. The attractor disappears at $a \approx 1.064$ after the boundary crisis, i. e. the contact with the basin of infinity (see Grebogi, *et al.*, 1982, Nusse & Yorke, 1994).

Let us now fix $b = 0.6$ and $a = 0.8$. This parameter point is inside the 8-tongue of the bifurcation diagram shown in Figure 4. The only attractor of the map F is

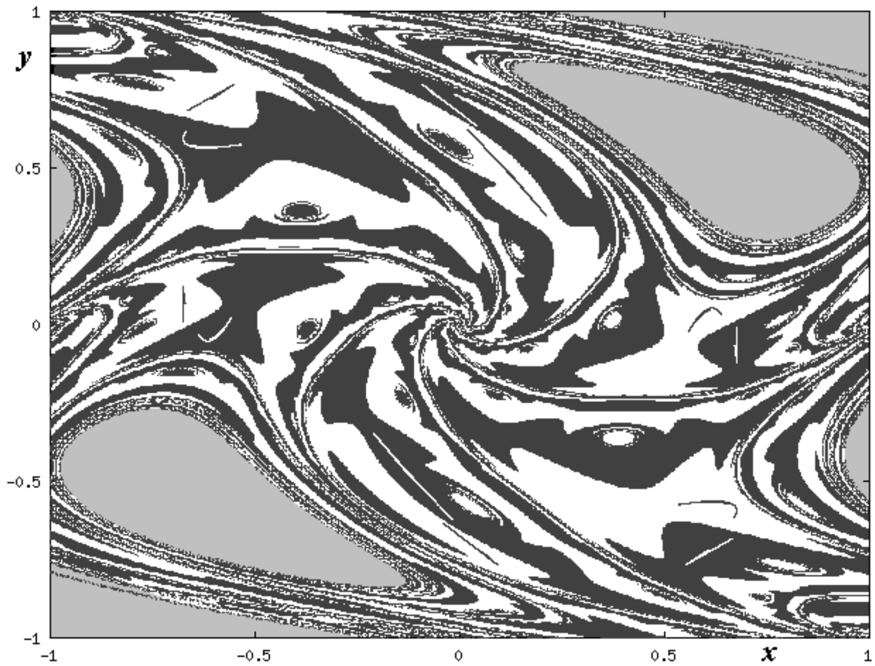

Figure 8: The two period-6 chaotic attractors and their basins of attraction at $a = 0.755$, $b = 1$.

the attracting period-8 cycle. The unstable manifold of the period-8 saddle cycle forms an invariant circle (see Figure 9).

We will increase a in a way shown in Figure 4 along the straight line with an arrow. At $a \approx 0.935$ the eigenvalues of the attracting period-8 cycle become complex-conjugate and thus this cycle becomes an attracting period-8 focus.

At $a \approx 1.02$ a pitchfork bifurcation of the period-8 saddle cycle occurs: this period-8 saddle becomes a repellor (a repelling node) but two new period-8 saddle cycles appear (see Figure 10).

At $a \approx 1.101$ a subharmonic saddle-node bifurcation occurs resulting in attracting and saddle cycles of the period 24, the points of which surround the points of the attracting period-8 focus by clusters of three. After this bifurcation there are two attracting sets: the period-8 focus, and the period-24 cycle, whose basins of attraction are shown in Figure 11. The immediate basin of attraction of a point of the attracting period-24 cycle is bounded by the stable manifold of the saddle period-24 cycle (we call it the 1^{st} 24-saddle). The boundary of infinity is formed, as in the previous examples, by the stable manifold of a saddle period-2 cycle.

We continue to increase the value of a. At $a \approx 1.118$ the attracting period-24 cycle undergoes a pitchfork bifurcation (its eigenvalue passes through 1) after which this cycle becomes a saddle (called by the 2^{nd} 24-saddle) and two new attracting 24-cycles are born. Now there are three attractors: the period-8 focus, and two period-24 cycles. An enlarged part of the phase space in this case is presented in Figure 12, where it can be seen that the stable manifold of the 1^{st} 24-saddle (the black squares) forms the immediate basin boundary of the attracting period-8 focus, while the stable manifold of the 2^{nd} 24-saddle (the white squares) forms the immediate basin boundary of two attracting period-24 cycles. Such a structure, in which the local stable manifold is part of a closed invariant curve, and approaches the saddle point from both sides, is made possible due to noninvertibility of the map F.

Next transformations of the phase portrait are connected to cascades of the period doubling bifurcations of both attracting period-24 cycles, resulting in two period-24 chaotic attractors. Thus, for instance, at $a = 1.125$ there are three attract-

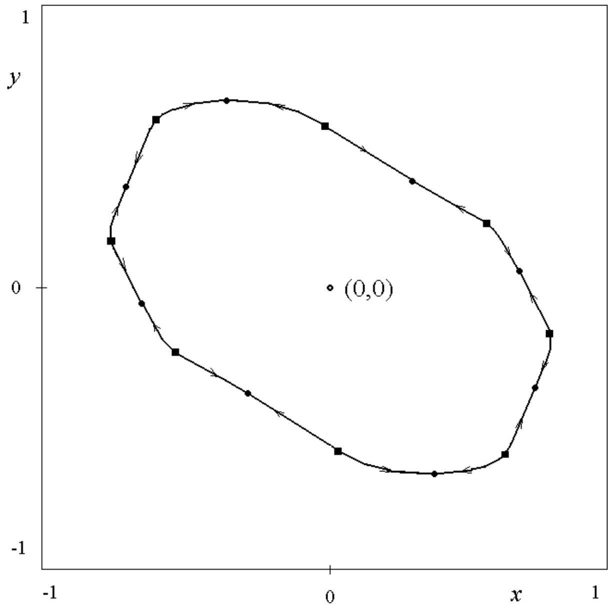

Figure 9: The invariant circle with the rotation number 1/8 of the map F at $a = 0.8$, $b = 0.6$. The points of the saddle period-8 cycle are shown by squares and of the attracting period-8 cycle by circles.

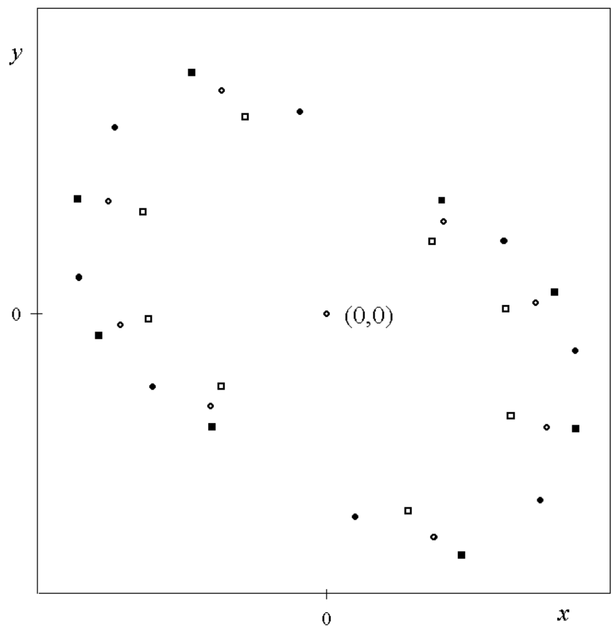

Figure 10: The phase portrait of the map F at $a = 1.03$, $b = 0.6$. The points of the two saddle period-8 cycles are shown by squares (black and white); the points of the repelling period-8 cycle are white circles and of the attracting period-8 cycle are black circles.

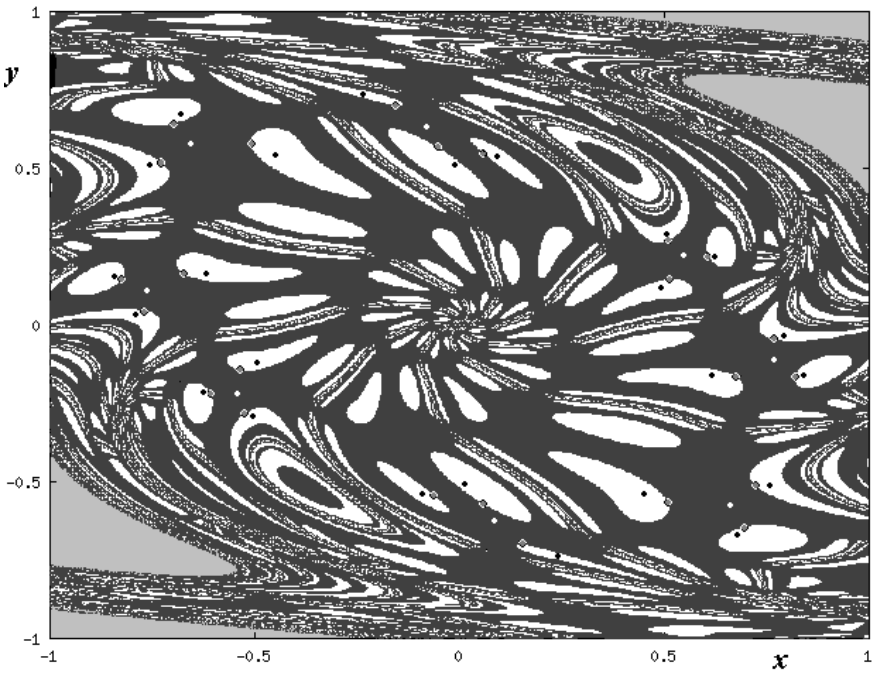

Figure 11: The attracting period-8 focus (the white circles) and the attracting period-24 cycle (the black circles) with their basins of attraction. The immediate basin boundary is formed by the stable manifold of the 1^{st} 24-saddle (the grey circles). Here $a = 1.102$, $b = 0.6$.

ors: the attracting period-8 focus and two period-24 chaotic attractors (see Figure 13).

The pairwise merger of the pieces of two chaotic attractors occurs at $a \approx 1.126$, due to the contact with the stable manifold of the 2^{nd} 24-saddle (which is the homoclinic bifurcation of this saddle). There are two attractors now: the period-8 focus and the period-24 chaotic attractor (see Figure 14).

Then at $a \approx 1.1358$ the period-24 chaotic attractor has a contact with the stable manifold of the 1^{st} 24-saddle. In other words, a homoclinic bifurcation of the 1^{st} 24-saddle occurs. As a result of this bifurcation, the period-24 chaotic attractor disappears, but the remnant of it, the period-8 invariant chaotic hyperbolic set, called chaotic saddle, still exists. This set can be seen as a transient trajectory soon after the bifurcation. The only attractor of the map F now is the period-8 focus, each point of which is surrounded by such a complicated set as the chaotic saddle.

To describe next bifurcations, let us argue in terms of the 8-th iteration of the map F, i. e. the map F^8, for which the period-8 focus is a fixed point (focus), the period-24 saddle is the period-3 saddle, and so on. At $a \approx 1.1451$ the attracting focus loses its stability via a Neimark-Sacker bifurcation (its complex eigenvalues pass through the unit circle), producing an invariant circle in its neighborhood (see Figure 15). As we increase a, the invariant circle moves nearer to the 1^{st} 3-saddle. Then at $a \approx 1.14574$ one branch of the stable manifold and one branch of the unstable manifold of the 1^{st} 3-saddle merge with the invariant circle, and this circle disappears, but the period-3 chaotic saddle reveals itself becoming a period-3 chaotic attractor. Figure 16 shows this bifurcation schematically. Figure 17 presents the resulting attractor for the map F.

Finally, at $a \approx 1.154$, due to the contact of the period-8 chaotic attractor with the stable manifold of the period-8 saddle (which forms the immediate basin boundary of each piece of the attractor), a one piece chaotic attractor appears. Leaving apart the transient bifurcations when cycles of other periods become attracting (see Figure 4) we show in Figure 18 the one-piece chaotic attractor near the boundary

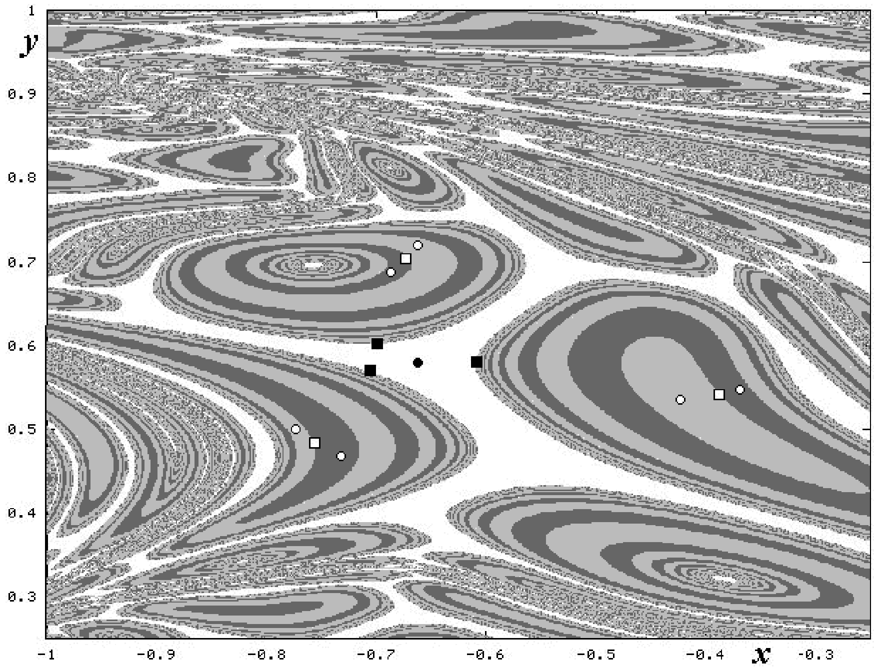

Figure 12: The immediate basin boundary of the period-8 focus is formed by the stable manifold of the 1^{st} 24-saddle (the black squares); the stable manifold of the 2^{nd} 24-saddle (the white squares) forms the immediate basin boundary of the attracting period-24 cycles.

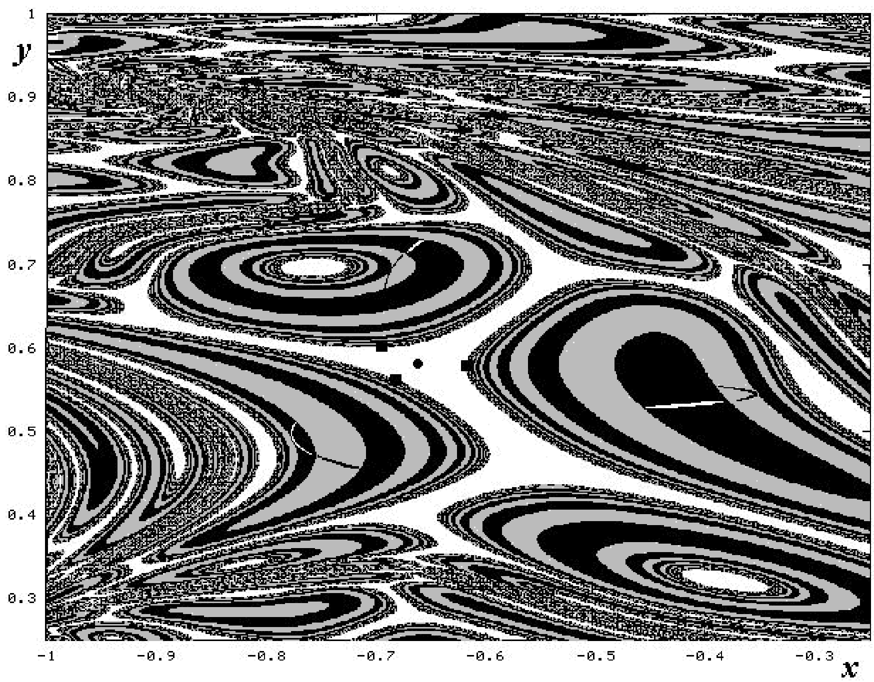

Figure 13: A part of the phase space where a point of the attracting period-8 focus (the black circle) and by three pieces of each period-24 chaotic attractor are shown together with the basins of attraction. Here $a = 1.125$, $b = 0.6$.

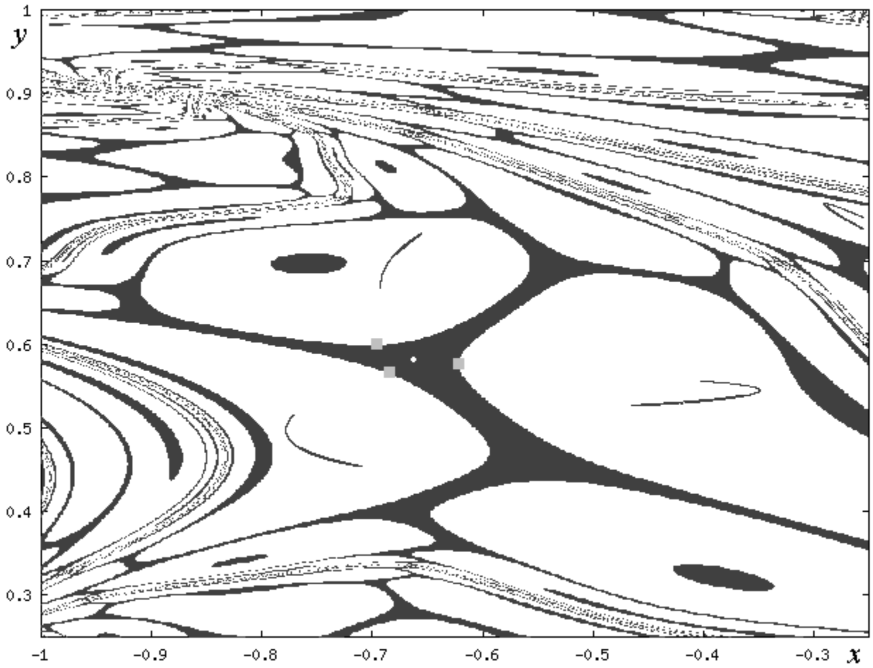

Figure 14: A part of phase space with one point of the attracting period-8 focus and three pieces of the period-24 chaotic attractor with the basins of attraction at $a = 1.126$, $b = 1$.

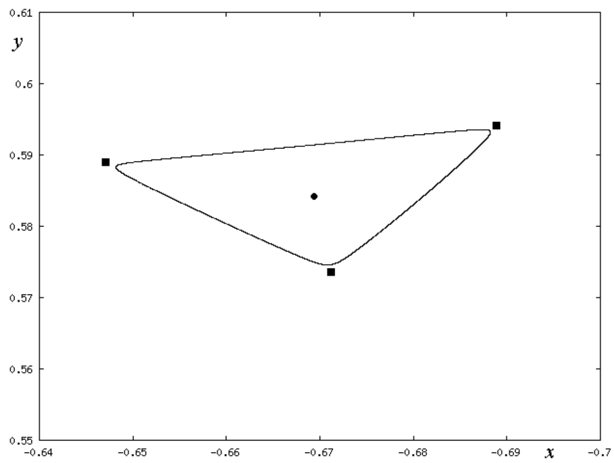

Figure 15: An invariant circle, the repelling focus (the black circle) and the 1^{st} 3-saddle (the black squares) of the map F^8 at $a = 1.1457$, $b = 0.6$.

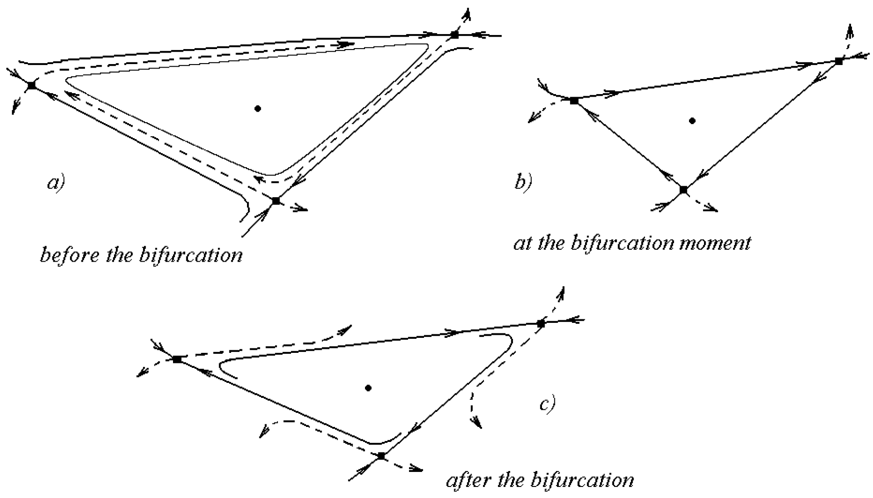

Figure 16: The sketch of the bifurcation of destruction of the invariant circle (the thin line in a)). The 1^{st} 3-saddle indicated by black squares, its unstable manifold by dotted lines and stable by thick lines.

crises, which occurs at $a \approx 1.471$.

Thus, we have described two examples of successive transformations of the phase portrait of the system under parameter variation. Taking the initial parameter point inside other tongues of periodicity, one can observe similar bifurcation scenarios as well as other interesting consequences of bifurcations.

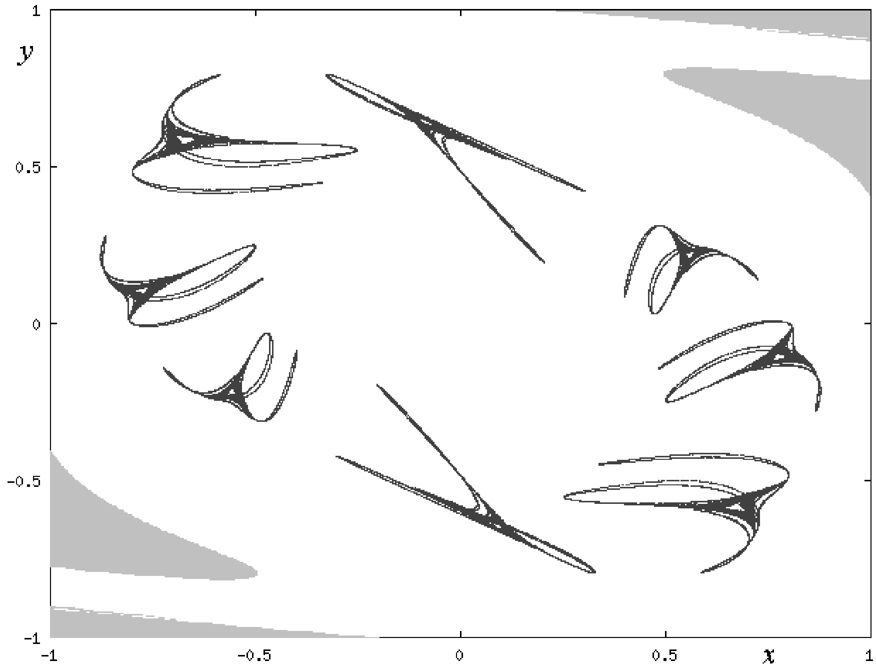

Figure 17: The period-8 chaotic attractor of the map F at $a = 1.148$, $b = 0.6$.

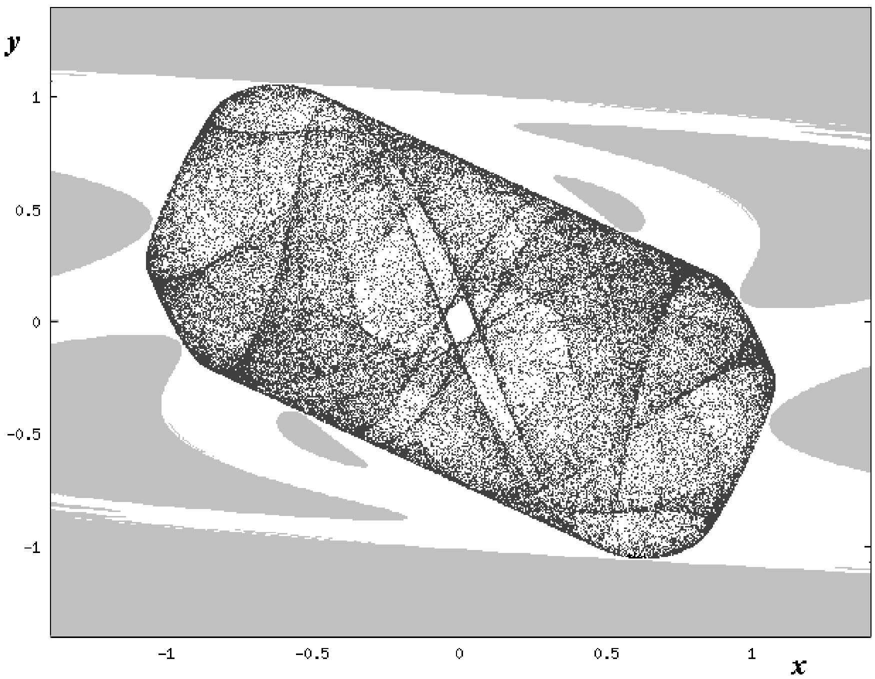

Figure 18: The chaotic attractor of the map F near the boundary crises at $a = 1.46$, $b = 0.6$.

Discussion

The setup of the model discussed was symmetric. As there is no factual need whatever for the "floor" and "roof" of the Hicksian investment function to be located at equal distances below and above the zero line, this case is not completely general. Hence, we should introduce some asymmetry through introducing a quadratic term along with the linear and cubic already present. Most certainly then several conclusions are changed when symmetry is broken. This would provide a point of departure for further study, in particular as we also get one more parameter associated with the quadratic term.

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ISBN 91-7305-289-2
ISSN 1404–5362